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THE RHEOLOGY OF SUSPENSIONS AND ITS RELATION TO PHENOMENOLOGICAL THEORIES FOR NON-NEWTONIAN FLUIDS

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Abstract—Theoretical results from the field of suspension rheology are studied in the general context of nonlinear continuum mechanics, in order to extract information regarding the formulation of a phenomenological stress relation to model non-Newtonian fluids. The constitutive equations obtained for dilute suspensions of solid ellipsoids, elastic spheres and liquid droplets are shown to conform to a single phenomenological model first proposed by Hand, which emerges as an equation with considerable physical basis and potential usefulness. Furthermore, in all of the above three cases, certain coefficients in Hand's equation are found to retain the same sign. This result may apply more generally and possibly have some interesting implications.

INTRODUCTION

The frequent occurrence of particulate systems in nature and in industrial processes makes it very desirable to achieve a good understanding of their flow properties and rheology, since such a knowledge can have important implications in various branches of engineering. In addition to this pragmatic aspect though, the study of suspensions is of considerable theoretical value. First of all, from a purely academic point of view, it gives rise to some fundamental fluid mechanical problems, many of which have remained unsolved. Another facet of the subject concerns certain non-Newtonian properties of flowing suspensions, such as a strain-rate dependent viscosity, normal stress effects and relaxation effects, which arise even when the disperse phase is very dilute. This remarkable behavior of particulate systems was recognized by numerous authors, e.g. Jeffery (1922) for solid ellipsoids, Batchelor (1970b) for slender solid particles, Lin, Peery & Schowalter (1970) for solid spheres under the influence of small but non-zero inertia forces, Oldroyd (1953, 1958) for liquid droplets, and Goddard & Miller (1967) and Roscoe (1967) for elastic spheres.

All the studies mentioned above dealt with cases in which the length scale of the motion is much larger than the dimensions of the particles, so that the suspension can be effectively considered as a continuum with bulk properties that are ensemble averages of the corresponding local quantities. Under these conditions then it is possible in principle, after obtaining the detailed flow field around each particle, to derive without any further assumptions an exact analytical rheological equation of state, which contains no adjustable parameters

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and in which the functional relation between the stress and all the relevant physical quantities is shown explicitly.

One of the drawbacks of this approach, to be termed suspension rheology, results from the fact that the analysis must be repeated for each particular suspension of interest. Evidently this is a serious inconvenience, because the microscopic problem is in general very difficult or even impossible to solve. However, since many suspensions as well as many non-Newtonian fluids do exhibit the same peculiar properties, one is led to believe (or hope) that their rheological behavior could be described by a general constitutive equation which would represent a large class of non-Newtonian continua.

A number of such general stress–strain relations have already been derived from purely phenomenological considerations, and familiar examples include, to name but a few, the Reiner–Rivlin equation, the Rivlin–Ericksen (1955) equation, Oldroyd's (1950) equations, and the equations for anisotropic fluids developed by Ericksen (1960a, b) and by Hand (1962). However, since these relationships are often obtained in a purely formal way, they are of rather limited usefulness from the practical point of view in that, because of their generality, they often contain a large number of unknown coefficients which cannot be determined using the presently available experimental techniques. Besides, since their derivation does not usually take into consideration the physical properties or the structure of the fluid, it is almost impossible to decide *a priori* which one, if any, of the existing phenomenological equations will apply for a given material.

Although, to date, there has been little interaction between suspension rheology and phenomenological theories, it is our belief that a great deal of information could be gained by considering concurrently the two approaches. Specifically, a comparison between the available phenomenological equations and the constitutive relations obtained for dilute suspensions of solid or deformable particles, could determine which, among the former are physically realistic, and can yield information regarding the physical significance of some of the phenomenological coefficients. Conversely, once this classification of the phenomenological equations is established, this new knowledge could be used to postulate *a priori* the form of the constitutive relation of a given suspension, and thus either to simplify the derivation of the stress–strain relation or to generalize the latter if it is known only for a particular case.

From those considerations, it is clear then that suspension rheology can play a central role in the formulation of a general phenomenological constitutive equation, and hence it is important that the subject be studied further. Indeed, a start in this direction has already been made by Gordon & Schowalter (1972). Here, we shall focus on one aspect of this topic, specifically the comparison between certain phenomenological equations and the actual stress–strain relations describing various suspensions, and shall show that the expression for the bulk stress of dilute suspensions of solid ellipsoids, elastic spheres or liquid droplets conforms to only one phenomenological constitutive relation, the one first proposed by Hand (1962), which thus emerges as an equation with considerable physical basis and potential usefulness. We further illustrate this comparison by proposing an extension to general shear flows, the constitutive equation obtained in the case of a simple shear flow, by Lin, Peery & Schowalter (1970) for a dilute suspension of solid spheres, under the influence of small but non-zero inertia forces.

PHENOMENOLOGICAL CONSTITUTIVE EQUATIONS FOR NON-NEWTONIAN FLUIDS

As was mentioned briefly in the Introduction, a large number of phenomenological constitutive equations has already been proposed which, it was hoped, could adequately describe the rheological behavior of non-Newtonian fluids. Of these the earliest was obtained by forming the most general relationship between the stress and the rate of strain when the former is assumed to depend only on the latter at time t, and gave rise to the Reiner–Rivlin (or Stokesian) fluid. Another model was proposed by Oldroyd (1950) which included time effects and resulted in the constitutive equation

$$(1 + \lambda(\mathscr{D}/\mathscr{D}t))P_{ij} - 2\kappa(e_{ik}P_{kj} + P_{ik}e_{ki}) = 2\eta(1 + \mu(\mathscr{D}/\mathscr{D}t))e_{ij} - 8\eta\nu e_{ik}e_{kj}, \qquad [1]$$

where the Cartesian tensor notation is adopted, P_{ij} is the stress tensor, e_{ij} is the rate of strain tensor, and λ, κ, μ, η and ν are constants characteristic of the material. Also, $\mathscr{D}/\mathscr{D}t$ denotes the Jaumann derivative, defined for an arbitrary tensor T_{ij}, by:

$$\frac{\mathscr{D}T_{ij}}{\mathscr{D}t} = \frac{\partial T_{ij}}{\partial t} + u_l\frac{\partial T_{ij}}{\partial x_l} + \tfrac{1}{2}\omega_l(\varepsilon_{ilm}T_{mj} + \varepsilon_{jlm}T_{mi}),$$

where u_l and ω_l are, respectively, the local velocity and vorticity of the fluid.

A more general theory was also proposed by Rivlin & Ericksen (1955), who postulated that the stress could be represented as a polynomial function of the rate of strain and its first N co-rotational (or Jaumann) derivatives, thereby yielding

$$P_{ij} = \alpha_0\delta_{ij} + \alpha_1 e_{ij} + \alpha_2\frac{\mathscr{D}e_{ij}}{\mathscr{D}t} + \alpha_3 Sd(e_{il}e_{lj}) + \alpha_4 Sd\left(e_{il}\frac{\mathscr{D}e_{lj}}{\mathscr{D}t}\right) + \alpha_5\frac{\mathscr{D}^2 e_{ij}}{\mathscr{D}t^2} + \alpha_6 Sd\left(\frac{\mathscr{D}e_{il}}{\mathscr{D}t}\frac{\mathscr{D}e_{lj}}{\mathscr{D}t}\right)$$

$$+ \alpha_7 Sd\left(e_{il}e_{lm}\frac{\mathscr{D}e_{mj}}{\mathscr{D}t}\right) + \alpha_8 Sd\left(e_{il}\frac{\mathscr{D}e_{lm}}{\mathscr{D}t}\frac{\mathscr{D}e_{mj}}{\mathscr{D}t}\right) + \alpha_9 Sd\left(\frac{\mathscr{D}e_{il}}{\mathscr{D}t}\frac{\mathscr{D}^2 e_{lj}}{\mathscr{D}t^2}\right), \qquad [2]$$

where the symbol Sd refers to the symmetric and traceless part of the indicated tensor, and the coefficients $\alpha_0, \ldots, \alpha_9$ are arbitrary functions of the simultaneous scalar invariants of the matrices corresponding to e_{ij}, $\mathscr{D}e_{ij}/\mathscr{D}t, \ldots \mathscr{D}^N e_{ij}/\mathscr{D}t^N$ and their products.

A difficulty, inherent in those theories, lies in the arbitrariness of the form selected for the constitutive equation and its apparent lack of physical justification. Furthermore, the proposed relationships are generally either algebraically simple but inaccurate when applied to real physical systems (e.g. the Reiner–Rivlin equation), or, if more complex, they contain a large number of unknown coefficients which cannot be uniquely determined with the present experimental techniques (e.g. the Rivlin–Ericksen equation).

In contrast, the theory of anisotropic fluids, first presented by Ericksen (1960a, b) has the advantage of taking into account the microscopic structure of the continuum. Indeed, a basic assumption is that the fluid is characterized at each point by a single preferred direction n_i and that the stress depends not only on the velocity gradient $\partial u_i/\partial x_j$ but also on the local anisotropy. Consequently,

$$P_{ij} = -p\delta_{ij} + f_{ij}(n_k \, . \, \partial u_k/\partial x_l),$$

where f_{ij} is an arbitrary function, symmetric in i and j, and p is an isotropic pressure term defined up to an additive constant in view of the assumed incompressibility of the fluid. The vector n_i is then supposed to satisfy a differential equation of the type:

$$\dot{n}_i = g_i(n_k, \partial u_k/\partial x_l),$$

where the dot denotes the co-rotational derivative.

By supposing that the functions f_{ij} and g_i are polynomials, and by applying the principle of material frame indifference, Ericksen (1960b) then obtained two closed-form expressions for the stress and for n_i, which, with the further restriction of a linear dependence on the rate of strain, become:

$$P_{ij} = -p\delta_{ij} + 2\mu e_{ij} + (\mu_1 + \mu_2 e_{lm}n_l n_m)n_i n_j + 2\mu_3(e_{il}n_l n_j + e_{jl}n_l n_i), \quad [3]$$

with

$$\mathscr{D}n_i/\mathscr{D}t = \lambda(e_{il}n_l - e_{lm}n_l n_m n_i), \quad [4]$$

where λ and the μ's are material constants. Evidently, since in the above the vector n_i is undistinguishable from $-n_i$, (3) and (4) seem particularly well suited for describing the stress of a dilute suspension of particles of revolution having a fore-and-aft symmetry.

As shown by Hand (1962), Ericksen's theory can easily be extended by assuming that the structure of the fluid is characterized at each point by a symmetric second-order tensor, A_{ij}. Thus, a reasoning identical to Ericksen's (1960b) leads to:

$$\begin{aligned} P_{ij} = {} & \beta_0\delta_{ij} + \beta_1 e_{ij} + \beta_2 Sd(A_{ij}) + \beta_3 Sd(e_{il}A_{lj}) + \beta_4 Sd(A_{il}A_{lj}) + \beta_5 Sd(e_{il}e_{lj}) \\ & + \beta_6 Sd(e_{il}A_{lm}A_{mj}) + \beta_7 Sd(e_{il}e_{lm}A_{mj}) + \beta_8 Sd(e_{il}e_{lm}A_{mp}A_{pj}), \end{aligned} \quad [5]$$

with

$$\begin{aligned} \frac{\mathscr{D}A_{ij}}{\mathscr{D}t} = {} & \gamma_0\delta_{ij} + \gamma_1 e_{ij} + \gamma_2 Sd(A_{ij}) + \gamma_3 Sd(e_{il}A_{lj}) + \gamma_4 Sd(A_{il}A_{lj}) + \gamma_5 Sd(e_{il}e_{lj}) \\ & + \gamma_6 Sd(e_{il}A_{lm}A_{mj}) + \gamma_7 Sd(e_{il}e_{lm}A_{mj}) + \gamma_8 Sd(e_{il}e_{lm}A_{mp}A_{pj}), \end{aligned} \quad [6]$$

where the β's and γ's are functions of the complete set of scalar invariants:

$$A_{il},\ A_{lm}A_{ml},\ A_{lm}A_{mp}A_{pl},\ e_{lm}e_{lm},\ e_{lm}e_{mp}e_{pl},\ e_{lm}A_{lm},\ e_{lm}e_{mp}A_{pl},\ e_{lm}A_{mp}A_{pl},\ e_{lm}e_{mp}A_{pq}A_{ql}. \quad [7]$$

Furthermore, if the stress is assumed to be linear in e_{ij}, [5] simplifies to:

$$\begin{aligned} P_{ij} = {} & (\sigma_0 + \sigma_1 e_{lm}A_{ml} + \sigma_2 e_{lm}A_{mp}A_{pl})\delta_{ij} + (\sigma_3 + \sigma_4 e_{lm}A_{ml} + \sigma_5 e_{lm}A_{mp}A_{pl})Sd(A_{ij}) \\ & + \sigma_6 e_{ij} + \sigma_7 Sd(e_{il}A_{lj}) + \sigma_8 Sd(e_{il}A_{lm}A_{mj}) \\ & + (\sigma_9 + \sigma_{10}e_{lm}A_{ml} + \sigma_{11}e_{lm}A_{mp}A_{pl})Sd(A_{il}A_{lj}), \end{aligned} \quad [8]$$

where the σ's are now functions only of the first three terms of [7]. Similarly, when $\mathscr{D}A_{ij}/\mathscr{D}t$

is also linear in e_{ij}. [6] becomes:

$$\mathscr{D}A_{ij}/\mathscr{D}t = (\theta_0 + \theta_1 e_{lm}A_{lm} + \theta_2 e_{lm}A_{mp}A_{pl})\delta_{ij} + (\theta_3 + \theta_4 e_{lm}A_{ml} + \theta_5 e_{lm}A_{mp}A_{pl})Sd(A_{ij}) + \theta_6 e_{ij} + \theta_7 Sd(e_{il}A_{lj}) + \theta_8 Sd(e_{il}A_{lm}A_{mj}) + (\theta_9 + \theta_{10}e_{lm}A_{ml} + \theta_{11}e_{lm}A_{mp}A_{pl})Sd(A_{il}A_{lj}). \quad [9]$$

It is obvious that [8] and [9] reduce, respectively, to [3] and [4] if the tensor A_{ij} is replaced by n_in_j.

The great advantage of this theory over most of the other phenomenological models lies in the fact that, in principle, it allows one to decide *a priori* whether or not such equations could adequately describe a given fluid of known physical properties. For example, it is evident that the constitutive equation of a dilute suspension of solid ellipsoids in the absence of Brownian motion should be expected to have the general form [5]. Of course, the restriction to an ellipsoidal anisotropy limits the applicability of Hand's relations to a special class of fluids, but this is hardly a fundamental difficulty since the theory can easily be extended by including higher-order tensors in the description of the microstructure of the material. Although the resulting equations become then more complicated and more difficult to handle, such a generalization will be seen to arise naturally in the case of dilute emulsions.

Let us consider at this point to what extent the fluids described by the various phenomenological equations mentioned above can be incorporated within the framework of Noll's theory. According to Coleman, Markovitz & Noll (1966) the stress of a "simple fluid" depends, at any instant t, only on the density ρ, and on the past history of motion U_{lm}, i.e.

$$P_{ij}(t) = -p\delta_{ij} + \mathop{f_{ij}}_{s=0}^{\infty}[U_{lm}(t-s), \rho(t)], \quad [10]$$

where f_{ij} is an isotropic, tensor-valued tensor functional. It is clear that Oldroyd's and Rivlin–Ericksen equations conform to this model and that, consequently, the materials they represent are simple fluids. However, such a conclusion cannot be drawn quite so readily for an anisotropic fluid, and a bit of analysis is required. We note first of all that Hand's relations can be expressed as

$$P_{ij} = -p\delta_{ij} + f_{ij}(e_{lm}, A_{lm}), \quad [11]$$

and

$$\mathscr{D}A_{ij}/\mathscr{D}t = g_{ij}(e_{lm}, A_{lm}), \quad [12]$$

and consider the case where the fluid, at rest until a time $t = t_0$, is subjected to a straining motion $e_{ij}(t)$ for $t > t_0$. Denoting by $\overset{(0)}{A}_{ij}(X)$ the anisotropy of the fluid at rest (e.g. the value of A_{ij} at $t = t_0$) due to particle X, we have for an incompressible substance that

$$A_{ij}(X, t) = G_{ij}\left[\mathop{e_{lm}}_{s=0}^{\infty}(X, t-s), \overset{(0)}{A}_{lm}(X)\right],$$

and therefore, because of [11], that

$$P_{ij} = p\delta_{ij} + f_{ij}\left\{e_{lm}, G_{lm}\left[\overset{\infty}{\underset{s=0}{e_{pq}}}(X, t-s), \overset{(0)}{A}_{pq}(X)\right]\right\}.$$

or that

$$P_{ij} = -p\delta_{ij} + F_{ij}\left[\overset{\infty}{\underset{s=0}{e_{lm}}}(X, t-s), \overset{(0)}{A}_{lm}(X)\right]. \qquad [13]$$

From a comparison of (10) and (13), it then follows that the fluids modeled by Hand's relations are not, in general, "simple fluids". However, if the anisotropy arises from the motion only, i.e. if the fluid is isotropic at rest, then the tensor $\overset{(0)}{A}_{lm}$ becomes independent of position and equals δ_{lm} or 0, depending on whether it measures the local anisotropy or the local deviation from isotropy. Consequently, in this instance, [13] reduces to [10],

It is clear, then, that suspensions of particles which are isotropic in a stress-free state, such as liquid droplets or elastic spheres, will be "pseudo-anisotropic fluids" and, hence, it should be possible to describe them with a constitutive equation similar to [10], provided a solution to the differential equation [12] can be obtained.

In the following sections, the rheology of several types of dilute suspensions will be studied and compared to some phenomenological theories. In all cases, the suspending medium will be taken as an incompressible Newtonian fluid of viscosity μ_0, and the disperse phase will be assumed sufficiently dilute for interactions between particles to be negligible. Inertia effects will be neglected except in one instance, where these will be given special consideration. The volume concentration of particles will be denoted by φ which, in our case, will be small compared with unity.

SUSPENSIONS OF DEFORMABLE PARTICLES

(a) *Liquid droplets*

The rheology of a dilute emulsion of two incompressible Newtonian liquids has been studied by many authors, for example by Schowalter, Chaffey & Brenner (1968) and by Frankel & Acrivos (1970). As illustrated by Frankel & Acrivos (1970), in cases where the deformation of the drops is small, a regular perturbation solution to the appropriate Stokes equations can be obtained for flow past a freely suspended drop, from which an expression for the bulk stress of the emulsion, as defined by Batchelor (1970a), follows readily. Two physical parameters of the suspension are of importance in this analysis: the ratio, λ, of the droplet viscosity to that of the ambient fluid, and the non-dimensional surface tension factor, k, defined as $\sigma/\mu_0 Ga$, where σ is the surface tension of the drop, μ_0 is the ambient viscosity, G is the magnitude of the shear flow, and a is the equivalent radius of the drop.

The constitutive relations can be derived for two limiting cases where the drop is kept nearly spherical on account either of its large surface tension ($k \gg 1$, $\lambda = O(1)$) or its high viscosity ($\lambda \gg 1$, $k = O(1)$). The results for the first case will now be examined.

By extending the earlier analysis of Frankel & Acrivos (1970) to higher orders in the drop deformation, Barthès-Biesel (1972) has shown that the bulk stress of a dilute emulsion is given by:

$$P_{ij} = -p\delta_{ij} + 2\mu_0 e_{ij} + \mu_0\varphi\left\{\left[\frac{10(\lambda-1)}{2\lambda+3} + \frac{24(\lambda-1)^2(611\lambda+579)}{49(2\lambda+3)^3}\beta^2 F_{lm}F_{lm}\right]e_{ij}\right.$$

$$+\left[\frac{24}{2\lambda+3} + \frac{240(\lambda-1)^2(121\lambda+159)}{49(2\lambda+3)^3}\beta^2 F_{lm}e_{lm}\right.$$

$$\left.-\frac{288(5912\lambda^3+48779\lambda^2+74931\lambda+29628)}{245(2\lambda+3)^3(19\lambda+16)}\beta^2 F_{lm}F_{lm}\right]F_{ij}$$

$$+\frac{360(\lambda-1)^2}{7(2\lambda+3)^2}\beta Sd(F_{il}e_{lj}) + \frac{288(\lambda-6)}{7(2\lambda+3)^2}\beta Sd(F_{il}F_{lj})$$

$$-\frac{720(\lambda-1)^2(79\lambda+96)}{49(2\lambda+3)^3}\beta^2 Sd(e_{il}F_{lm}F_{mj}) - \frac{800(\lambda-1)^2}{(2\lambda+3)^2}\beta^2 F_{ijlm}e_{lm}$$

$$+\frac{240(103020\lambda^4+481092\lambda^3+433959\lambda^2+549640\lambda+136576)}{(2\lambda+3)^2(19\lambda+16)(17\lambda+16)(10\lambda=11)}\beta^2 F_{ijlm}F_{lm}$$

$$\left.+O(Gk^{-3})\right\}, \qquad [14]$$

where $\beta = 1/Gk$. F_{ij} and F_{ijab} are symmetric tensors describing the shape of the drop the equation of which, in a coordinate system moving with the center of a particle, is:

$$\frac{r}{a} = (x_l x_l)^{1/2} = 1 + 3\beta F_{lm}\frac{x_l x_m}{r^2} + \beta^2\left(-\frac{6}{5}F_{lm}F_{lm} + 105F_{lmpq}\frac{x_l x_m x_p x_q}{r^4}\right) + O(Gk^{-3}).$$

These tensors are determined by the two differential equations

$$F_{ij} + \frac{(2\lambda+3)(19\lambda+16)}{40(\lambda+1)}\beta\frac{\mathscr{D}F_{ij}}{\mathscr{D}t}$$

$$= \left[\frac{19\lambda+16}{24(\lambda+1)} - \frac{(11172\lambda^4+18336\lambda^3+17440\lambda^2+3499\lambda-7572)}{980(2\lambda+3)^2(\lambda+1)}\beta^2 F_{lm}F_{lm}\right]e_{ij}$$

$$-\left[\frac{(\lambda-1)(22344\lambda^3+52768\lambda^2+45532\lambda+19356)}{980(2\lambda+3)^2(\lambda+1)}F_{lm}e_{lm}\right.$$

$$\left.+\frac{6C_7(\lambda)F_{lm}F_{lm}}{245(2\lambda+3)^2(19\lambda+6)^2(10\lambda+11)(17\lambda+16)(\lambda+1)}\right]\beta^2 F_{ij}$$

$$+\frac{(4\lambda-9)(19\lambda+16)}{28(2\lambda+3)(\lambda+1)}\beta Sd(e_{il}F_{lj}) + \frac{36(137\lambda^3+624\lambda^2+741\lambda+248)}{35(2\lambda+3)(19\lambda+16)(\lambda+1)}\beta Sd(F_{il}F_{lj})$$

$$+\frac{6(\lambda-1)(2793\lambda^3+7961\lambda^2+8474\lambda+3522)}{245(2\lambda+3)^2(\lambda+1)}\beta^2 Sd(e_{il}F_{lm}F_{mj})$$

$$-\frac{10(43\lambda^2+79\lambda+53)}{3(2\lambda+3)(\lambda+1)}\beta^2 F_{ijlm}e_{lm}+\frac{2C_5(\lambda)\beta^2 F_{ijlm}F_{lm}}{(2\lambda+3)(19\lambda+16)(10\lambda+11)(17\lambda+16)(\lambda+1)}$$

$$+O(Gk^{-3}), \qquad [15]$$

and

$$F_{ijab}+\frac{(17\lambda+16)(10\lambda+11)}{360(\lambda+1)}\beta\frac{\mathscr{D}F_{ijb}}{\mathscr{D}t}=\frac{(17\lambda+16)(10\lambda+11)}{2520(\lambda+1)(2\lambda+3)}Sd_4(e_{ij}F_{ab})$$

$$+\frac{2(-14\lambda^2+221\lambda+192)}{945(2\lambda+3)(19\lambda+16)}Sd_4(F_{ij}F_{ab})+O(Gk^{-1}), \qquad [16]$$

where

$$C_7(\lambda)=2127976\lambda^7-16341920\lambda^6-38494964\lambda^5+122942551\lambda^4$$
$$+474068311\lambda^3+591515680\lambda^2+332123136\lambda+71700480,$$

and

$$C_5(\lambda)=405260\lambda^5+2366960\lambda^4+9142173\lambda^3+8595967\lambda^2+3334160\lambda+693760.$$

It is apparent now by inspection that the expressions given above bear a striking similarity to Hand's equations provided A_{ij} is identified with F_{ij}. The appearance of higher-order tensors (i.e. of order higher than 2) is then due to the fact that the droplet does not remain ellipsoidal but instead, assumes a more complex shape. Still, [14], [15] and [16] clearly represent a possible extension of Hand's theory to include more general types of microscopic anisotropies.

It is important to note also that the physical analysis applied to the rheology of a dilute emulsion does yield two different sets of equations, as predicted by Hand: one expression relating the stress to the rate of strain and to a measure of the local anisotropy, and a set of differential equations describing the variation of the anisotropy as a function of time and of the rate of strain. Furthermore, all the terms appearing in Hand's phenomenological relations arise naturally from the analytical solution to the flow problem as already discussed by Frankel & Acrivos (1970).

Next, the unknown coefficients of [8] and [9] can easily be related to the physical parameters of the suspension. Thus, for the stress equation, the values of the σ's, to $O(Gk^{-3})$, are:

$$\sigma_0=-p, \qquad \sigma_1=\sigma_2=0,$$

$$\sigma_3=\mu_0\varphi\left[\frac{24}{2\lambda+3}-\frac{288(5912\lambda^3+48779\lambda^2+74941\lambda+29628)}{245(2\lambda+3)^3(19\lambda+16)}\beta^2(A_{lm}A_{lm})\right],$$

$$\sigma_4=\frac{240(\lambda-1)^2(121\lambda+159)}{49(2\lambda+3)^3}\beta^2\mu_0\sigma, \qquad \sigma_5=0,$$

$$\sigma_6=2\mu_0\left\{1+\varphi\left[\frac{5(\lambda-1)}{2\lambda+3}+\frac{12(\lambda-1)^2(611\lambda+579)}{49(2\lambda+3)^3}\beta^2(A_{lm}A_{lm})\right]\right\},$$

$$\sigma_7=\frac{360(\lambda-1)^2}{7(2\lambda+3)^2}\beta\mu_0\varphi, \qquad \sigma_8=-\frac{720(\lambda-1)^2(79\lambda+96)}{49(2\lambda+3)^3}\beta^2\mu_0\varphi,$$

$$\sigma_9=\frac{288(\lambda-6)}{7(2\lambda+3)^2}\beta\mu_0\varphi, \qquad \sigma_{10}=\sigma_{11}=0. \qquad [17]$$

In addition, since the differential equation for F_{ij} is, in this case, also linear in the rate of strain, a comparison between [9] and [15] yields the following expressions, to order (Gk^{-2}), for the coefficients θ:

$$\theta_0 = \theta_1 = \theta_2 = 0,$$

$$\theta_3 = \frac{-40(\lambda+1)}{(2\lambda+3)(19\lambda+16)}\beta^{-1} - \frac{48C_7(\lambda)(A_{lm}A_{lm})}{49(2\lambda+3)^3(19\lambda+16)^3(10\lambda+11)(17\lambda+16)}\beta,$$

$$\theta_4 = -\frac{2(\lambda-1)(22344\lambda^3+52768\lambda^2+45532\lambda+19356)}{49(2\lambda+3)^3(19\lambda+16)}\beta, \qquad \theta_5 = 0,$$

$$\theta_6 = \frac{5}{3(2\lambda+3)}\beta^{-1} - \frac{2(11172\lambda^4+18336\lambda^3+17440\lambda^2+3499\lambda-7572)}{49(2\lambda+3)^3(19\lambda+16)}\beta(A_{lm}A_{lm}),$$

$$\theta_7 = \frac{10(4\lambda-9)}{7(2\lambda+3)^2}, \qquad \theta_8 = \frac{48(\lambda-1)(2793\lambda^3+7961\lambda^2+8474\lambda+3522)}{49(2\lambda+3)^3(19\lambda+16)}\beta,$$

$$\theta_9 = \frac{288(137\lambda^3+624\lambda^2+741\lambda+248)}{7(2\lambda+3)^2(19\lambda+16)^2}, \qquad \theta_{10} = \theta_{11} = 0. \qquad [18]$$

Similarly, for the case of highly viscous drops, the same procedure as above yields up to order $(G\lambda^{-2})$ that:

$$\sigma_0 = -p, \qquad \sigma_1 = \sigma_2 = \sigma_5 = \sigma_9 = \sigma_{10} = \sigma_{11} = 0,$$

$$\sigma_3 = \frac{12}{\beta}\lambda^{-2}\mu_0\varphi, \qquad \sigma_4 = \frac{3630}{49}\lambda^{-2}\mu_0\varphi,$$

$$\sigma_6 = \mu_0\left\{2 + \varphi\left[5 - \frac{25}{2}\lambda^{-1} + \lambda^{-2}\left(\frac{75}{4} + \frac{1833}{49}A_{lm}A_{lm}\right)\right]\right\},$$

$$\sigma_7 = \left(\frac{90}{7}\lambda^{-1} - \frac{450}{7}\lambda^{-2}\right)\mu_0\varphi, \qquad \sigma_8 = -\frac{7110}{49}\lambda^{-2}\mu_0\varphi,$$

$$\theta_0 = \theta_1 = \theta_2 = \theta_5 = \theta_9 = \theta_{10} = \theta_{11} = 0,$$

$$\theta_3 = -\frac{20}{19}(\beta\lambda)^{-1}, \qquad \theta_4 = -6\lambda^{-1},$$

$$\theta_6 = \frac{5}{6} - \frac{5}{4}\lambda^{-1} - 3\lambda^{-2}A_{lm}A_{lm}, \qquad \theta_7 = \frac{10}{7}\lambda^{-1},$$

$$\theta_8 = 18\lambda^{-1}.$$

Although conforming to Hand's equations, a dilute emulsion is in fact a "pseudo-anisotropic" fluid, since its anisotropy is developed only as a consequence of the motion, and hence its constitutive equation should reduce to that of a simple fluid. Specifically, for weakly time-dependent flows, i.e. such at $\beta(\mathscr{D}/\mathscr{D}t)$ is $O(\beta)$, [15] and [16] can be solved by a method of successive approximations which yields:

$$F_{ij} = \frac{19\lambda + 16}{24(\lambda + 1)}\left\{e_{ij} + \beta\left[\frac{601\lambda^2 + 893\lambda + 256}{140(\lambda + 1)^2} Sd(e_{il}e_{lj}) - \frac{(2\lambda + 3)(19\lambda + 16)}{40(\lambda + 1)}\frac{\mathscr{D}e_{ij}}{\mathscr{D}t}\right]\right.$$
$$+ \beta^2\left[\frac{(19\lambda + 16)^2(2\lambda + 3)^2}{1600(\lambda + 1)^2}\frac{\mathscr{D}^2 e_{ij}}{\mathscr{D}t^2}\right.$$
$$- \frac{3(19\lambda + 16)(1476\lambda^3 + 4837\lambda^2 + 4673\lambda + 1264)}{5600(\lambda + 1)^3} Sd\left(e_{il}\frac{\mathscr{D}e_{lj}}{\mathscr{D}t}\right)$$
$$\left.\left. + \frac{P_7(\lambda)}{(2\lambda + 3)(17\lambda + 16)(10\lambda + 11)(\lambda + 1)^4} e_{ij}(e_{lm}e_{lm})\right]\right\} + O(G^4\beta^3),$$

and
$$F_{ijab} = \frac{751\lambda + 656}{544320(\lambda + 1)^2} Sd_4(e_{ij}e_{ab}) + O(G^2\beta).$$

Then the constitutive equation becomes:

$$P_{ij} = -p\delta_{ij} + 2\mu_0 e_{ij}\left\{1 + \varphi\left[\frac{5\lambda + 2}{2(\lambda + 1)} + \frac{(19\lambda + 16)R_7(\lambda)\beta^2}{(2\lambda + 3)(17\lambda + 16)(10\lambda + 11)(\lambda + 1)^5} e_{lm}e_{lm}\right]\right\}$$
$$+ \mu_0\varphi\beta\frac{(19\lambda + 16)}{20(\lambda + 1)^2}\left\{-\frac{(19\lambda + 16)}{2}\frac{\mathscr{D}e_{ij}}{\mathscr{D}t} + \frac{3(25\lambda^2 + 41\lambda + 4)}{7(\lambda + 1)} Sd(e_{il}e_{lj})\right.$$
$$- \beta\frac{(19\lambda + 16)(150\lambda^3 + 2179\lambda^2 + 2897\lambda + 724)}{280(\lambda + 1)^2} Sd\left(e_{il}\frac{\mathscr{D}e_{lj}}{\mathscr{D}t}\right)$$
$$\left. + \beta\frac{(19\lambda + 16)^2(2\lambda + 3)}{80(\lambda + 1)}\frac{\mathscr{D}^2 e_{ij}}{\mathscr{D}t^2}\right\} + O(G^4\beta^3), \qquad [19]$$

where
$$P_7(\lambda) = 3103.908\lambda^7 + 20684.86\lambda^6 + 72725.24\lambda^5 + 123993.98\lambda^4 + 103839.12\lambda^3 + 41745.30\lambda^2 + 7148.42\lambda + 428.53,$$

and
$$R_7(\lambda) = 422.28\lambda^7 + 3208.5\lambda^6 + 9960.91\lambda^5 + 11052.97\lambda^4 + 5115.57\lambda^3 + 2848.07\lambda^2 + 1711.66\lambda + 100.86.$$

It is immediately apparent that [19] is identical to [2], the stress relation given by Rivlin & Ericksen (1955) in their theory of isotropic fluids.

By neglecting terms of $O(\beta^2)$, Frankel & Acrivos (1970) obtained an expression identical to [19] and showed that their relation could then be recast in terms of Oldroyd's constitutive equation, as given by [1]. It is interesting to note, however, that, when the $O(\beta^2)$ terms are taken into account, such a transformation is no longer feasible owing to the occurrence of higher-order time derivatives of e_{ij} and cross products between e_{ij} and its derivatives.

(b) *Elastic spheres*

Goddard & Miller (1967) studied the rheology of a dilute suspension of elastic spheres. For a Hookean solid, and for small deformations of the particles, they obtained the constitutive equation:

$$P_{ij} = -p\delta_{ij} + 2\mu_0 e_{ij} + 5\mu_0\varphi\{e_{ij} - (\mathscr{D}D_{ij}/\mathscr{D}t) + \tfrac{6}{7}Sd(e_{il}C_{lj}) - \tfrac{20}{7}Sd(C_{il}(\mathscr{D}C_{lj}/\mathscr{D}t)) + O(G\varepsilon^2)\}, \quad [20]$$

and also showed that C_{ij}, the finite strain tensor of the particles, satisfies the differential equation:

$$\mathscr{D}C_{ij}/\mathscr{D}t + (1/\tau)C_{ij} = \tfrac{5}{3}e_{ij} + \tfrac{10}{7}Sd(e_{il}C_{lj}) - \tfrac{24}{7}Sd(C_{il}(\mathscr{D}C_{lj}/\mathscr{D}t)) + O(G\varepsilon^2), \quad [21]$$

where $\tau = 3\mu_0/2\kappa$, with κ being the elastic modulus of the particles. Furthermore, since the above apply only for small deformations, the magnitude of any element of C_{ij} is $O(\varepsilon)$, where ε is much smaller than unity and is defined by $\varepsilon = \tau G \ll 1$. Without loss of generality, [21] can be solved by successive approximations to eliminate cross products between C_{ij} and $\mathscr{D}C_{ij}/\mathscr{D}t$, and becomes

$$\mathscr{D}C_{ij}/\mathscr{D}t = \tfrac{5}{3}e_{ij} - (1/\tau)C_{ij} - \tfrac{30}{7}Sd(e_{il}C_{lj}) + (24/7\tau)Sd(C_{il}C_{lj}) + O(G\varepsilon^2), \quad [22]$$

and hence, the constitutive equation can be recast into:

$$P_{ij} = -p\delta_{ij} + 2\mu_0(1 - \tfrac{5}{3}\varphi)e_{ij} + 5\mu_0\varphi[(1/\tau)C_{ij} + \tfrac{8}{21}Sd(e_{il}C_{lj}) - (4/7\tau)Sd(C_{il}C_{lj}) + O(G\varepsilon^2)]. \quad [23]$$

Again, a comparison between Hand's relations [8] and [9] and those just derived for a dilute suspension of elastic spheres, shows that the two cases are equivalent if the anisotropy tensor A_{ij} is identified with the finite strain tensor C_{ij}. To be sure, this result is not surprising and could have been foreseen. Indeed, it can be shown that when an elastic sphere is suspended in a homogeneous shear field, the stress system inside the particle is also homogeneous so that the sphere deforms into an ellipsoid. Such a suspension assumes therefore the particular type of anisotropy considered in Hand's theory, and is naturally described by [8] and [9].

The unknown coefficients appearing in Hand's equations can now be readily determined to $O(G\varepsilon^2)$ and become

$$\sigma_0 = -p \text{ (arbitrary pressure term)}, \quad \sigma_3 = 5\mu_0\varphi/\tau, \quad \sigma_6 = 2\mu_0(1 - \tfrac{5}{3}\varphi),$$
$$\sigma_7 = \tfrac{40}{21}\mu_0\varphi, \quad \sigma_9 = -(20/7\tau)\mu_0\varphi, \quad \theta_3 = -(1/\tau), \quad \theta_6 = \tfrac{5}{3}, \quad \theta_7 = -\tfrac{30}{7}, \quad \theta_9 = 24/7\tau, \quad [24]$$

all other coefficients being zero.

Because of its isotropy at rest, a dilute suspension of elastic spheres is another example of a pseudo-anisotropic fluid, and thus, its constitutive equation is expected to reduce to that of a simple fluid. Again then, for weakly time-dependent flows and provided that τ is small compared with unity, a method of successive approximations will again yield a constitutive equation of the type proposed by Rivlin and Ericksen:

$$P_{ij} = -p\delta_{ij} + 2\mu_0\left(1 + \frac{5}{2}\varphi\right)e_{ij} + 25\mu_0\varphi\left[-\frac{\tau}{3}\frac{\mathscr{D}e_{ij}}{\mathscr{D}t} + \frac{2\tau}{7}Sd(e_{il}e_{lj}) + O(G^3\tau^2)\right]. \quad [25]$$

Although the above can also be recast in the form of Oldroyd's equation, it seems reasonable to assume, in view of the results obtained for an emulsion, that this transformation would no longer be feasible if the higher-order terms were included in [25].

It is worthwhile to note the important similarity existing between suspensions of elastic spheres and liquid droplets, which appears to result from the fact that the surface tension forces play qualitatively the same role as the elastic forces in resisting any further deformation of the particles comprising the suspension.

SUSPENSIONS OF RIGID PARTICLES

(a) *Solid ellipsoids*

Starting from Jeffery's (1922) expressions for the creeping flow field past a solid ellipsoid, Hand (1961b) derived a constitutive equation for a dilute suspension of such particles in the absence of Brownian motion which was also obtained by Batchelor (1970a) using a somewhat more general approach. In Batchelor's formulation, the bulk stress is defined as an ensemble average of the local microscopic stresses, over a representatative volume, V, enclosing N particles, and is expressed as

$$P_{ij} = -p\delta_{ij} + 2\mu_0 e_{ij} + \frac{4\pi\mu_0}{V}\sum D_{ij}, \quad [26]$$

where the summation is taken over all particles in V. Also, D_{ij} represents the contribution to the stress arising from the presence of the particles, and is easily derived from Jeffery's results.

Evidently, the theory of anisotropic fluids seems well suited to model such a suspension, since the local anisotropy due to the particles can easily be represented by a symmetric second-order tensor (e.g. the tensor of the surface of the ellipsoid). In particular, using his definition of the bulk properties of the suspension, Hand (1961b) established an exact correspondence between Ericksen's equations and those describing the rheology of a dilute suspension of spheroids, where, in this case, the preferred direction of the fluid can be chosen to coincide, at every instant, with the axis of revolution of the particles. Hand (1961a) also showed that his phenomenological equations could model a suspension of arbitrary ellipsoids, and gave the corresponding values for the coefficients in [8] and [9] when the anisotropy, A_{ij}, is the tensor of a particle.

Using Batchelor's definition of the bulk properties of a suspension of ellipsoids, in the case where V contains one particle, we shall now give the corresponding expression for those coefficients in [8] and [9] in a form which clearly brings out their symmetric dependence on the semi-diameters a, b, c of the ellipsoids. First, an orthonormal system of three unit vectors $\overset{(a)}{n_i}, \overset{(b)}{n_i}, \overset{(c)}{n_i}$ directed along the principal axes of one particle is chosen. Next, the tensor

A_{ij} is defined as

$$A_{ij} = \frac{1}{a^2}\overset{(a)(a)}{n_i n_j} + \frac{1}{b^2}\overset{(b)(b)}{n_i n_j} + \frac{1}{c^2}\overset{(c)(c)}{n_i n_j},$$

in a fixed Cartesian coordinate system, it being understood that A_{ij} is a function of time, since the ellipsoids are rotating. Then, by comparing [26] and [8], after replacing D_{ij} by its value given by Batchelor (1970a), we obtain

$$\sigma_0 = -p, \qquad \sigma_1 = \sigma_2 = \sigma_3 = \sigma_9 = 0,$$

$$\sigma_4 = -\frac{4\mu_0\varphi}{abcD}[A''a^8(b^4-c^4)^2+B''b^8(c^4-a^4)^2+C''c^8(a^4-b^4)^2+4A'b^4c^4(c^4-a^4)(a^4-b^4)$$
$$+ 4B'a^4c^4(a^4 - b^4)(b^4 - c^4) + 4C'a^4b^4(b^4 - c^4)(c^4 - a^4)],$$

$$\sigma_5 = \sigma_{10} = \frac{4(abc)\mu_0\varphi}{D}[A''a^6(b^2 + c^2)(b^2 - c^2)^2 + B''b^6(c^2 + a^2)(c^2 - a^2)^2$$
$$+ C''c^6(a^2 + b^2)(a^2 - b^2)^2 + 2A'b^2c^2(c^2 - a^2)(a^2 - b^2)(2b^2c^2 + c^2a^2 + a^2b^2)$$
$$+ 2B'c^2a^2(a^2 - b^2)(b^2 - c^2)(2a^2c^2 + c^2b^2 + b^2a^2)$$
$$+ 2C'a^2b^2(b^2 - c^2)(c^2 - a^2)(2a^2b^2 + b^2c^2 + c^2a^2)],$$

$$\sigma_6 = 2\mu_0 + \frac{8\mu_0\varphi}{abc}\left\{A'\left[1 + \frac{2a^4}{(c^2 - a^2)(a^2 - b^2)}\right] + B'\left[1 + \frac{2b^4}{(a^2 - b^2)(b^2 - c^2)}\right]\right.$$
$$\left.+ C'\left[1 + \frac{2c^4}{(b^2 - c^2)(c^2 - a^2)}\right]\right\},$$

$$\sigma_7 = \frac{16\mu_0\varphi}{abc}\left[\frac{A'a^4(b^4 - c^4) + B'b^4(c^4 - a^4) + C'c^4(a^4 - b^4)}{(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)}\right],$$

$$\sigma_8 = -16\mu_0\varphi(abc)\left[\frac{A'a^2(b^2 - c^2) + B'b^2(c^2 - a^2) + C'c^2(a^2 - b^2)}{(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)}\right],$$

$$\sigma_{11} = -4(abc)^3\mu_0\varphi D^{-1}[A''a^4(b^2 - c^2)^2 + B''b^4(c^2 - a^2)^2 + C''c^4(a^2 - b^2)^2$$
$$+ 4A'b^2c^2(c^2 - a^2)(a^2 - b^2) + 4B'c^2a^2(a^2 - b^2)(b^2 - c^2)$$
$$+ 4C'a^2b^2(b^2 - c^2)(c^2 - a^2)],$$

where $D = (a^2 - b^2)(b^2 - c^2)(c^2 - a^2)[b^2c^2(b^2 - c^2) + c^2a^2(c^2 - a^2) + a^2b^2(a^2 - b^2)]$.

Also, A', B', C' and A'', B'', C'' are related to the integrals $\alpha_0', \beta_0', \gamma_0'$ and $\alpha_0'', \beta_0'', \gamma_0''$, introduced by Jeffery (1922), by means of $A' = 1/2(b^2 + c^2)\alpha_0'$, with similar expressions for B' and C', and $A'' = \alpha_0''/(\alpha_0''\beta_0'' + \beta_0''\gamma_0'' + \gamma_0''\alpha_0'')$, with similar expressions for B' and C''. Similarly, from a comparison between Jeffery's equations of motion of the ellipsoid and [9], it is possible to show that the various coefficients θ become:

$$\theta_0 = \theta_1 = \theta_2 = \theta_3 = \theta_9 = 0,$$

$$\theta_4 = \frac{2[(a^2b^2 + b^2c^2 + c^2a^2)^2 + (abc)^2(a^2 + b^2 + c^2)]}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)},$$

$$\theta_5 = \theta_{10} = -\frac{2(abc)^2(a^2b^2 + b^2c^2 + c^2a^2)}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)},$$

$$\theta_6 = \frac{4[(a^2 + b^2 + c^2)^2 - (a^2b^2 + b^2c^2 + c^2a^2)]}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}, \qquad \theta_7 = 6,$$

$$\theta_8 = -\frac{8(abc)^2(a^2 + b^2 + c^2)}{(a^2 + b^2)(b^2 + c^2)(c^3 + a^2)}, \qquad \theta_{11} = \frac{2(abc)^4}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}.$$

The algebraic complexity of some of these expressions limits the practical interest of this result. However, the numerical values of the σ's and θ's were computed for a variety of shapes and are given in tables 1 and 2. It was found, interestingly enough, that those coefficients always retained the same sign, as shown in table 3. Their asymptotic behavior was also investigated for three particular shapes: an almost spherical ellipsoid, a slender ellipsoid, and a flat and almost circular ellipsoid. In all three cases, the product abc is set equal to unity which, of course, does not restrict the generality of the results, since this condition can be satisfied for any ellipsoid after a simple affine transformation. These asymptotic expansions should be useful in assessing the magnitude of the coefficients in Hand's equations for suspensions of arbitrary slender bodies or of disk-like bodies.

Table 1. Values of the coefficients in Hand's stress equation for a dilute suspension of solid ellipsoids, as a function of the shape of the particles.

$(1 - b^2/a^2)^{1/2}$	Eccentricity $(1 - c^2/b^2)^{1/2}$	σ^4	σ_5 or σ_{10}	σ_6	σ_7	σ_8	σ_{11}
0.2	0.2	8.147	−2.543	5.256	1.632	−1.886	0.842
	0.4	8.167	−2.541	5.265	1.631	−1.873	0.838
	0.6	8.294	−2.534	5.320	1.622	−1.805	0.819
	0.8	9.121	−2.545	5.673	1.581	−1.519	0.741
0.4	0.2	8.164	−2.540	5.262	1.631	−1.873	0.838
	0.4	8.194	−2.537	5.275	1.628	−1.853	0.832
	0.6	8.349	−2.530	5.342	1.617	−1.773	0.809
	0.8	9.279	−2.546	5.732	1.574	−1.473	0.727
0.6	0.2	8.238	−2.522	5.284	1.621	−1.807	0.817
	0.4	8.285	−2.516	5.301	1.616	−1.775	0.807
	0.6	8.489	−2.506	5.383	1.602	−1.676	0.778
	0.8	9.618	−2.531	5.832	1.552	−1.358	0.691
0.8	0.2	8.491	−2.421	5.306	1.574	−1.534	0.723
	0.4	8.558	−2.407	5.319	1.565	−1.489	0.707
	0.6	8.845	−2.388	5.410	1.543	−1.374	0.671
	0.8	10.353	−2.426	5.925	1.481	−1.072	0.583

Table 2. Values of the coefficients in Hand's anisotropy equation for a dilute suspension of solid ellipsoids, as a function of the shape of the particles.

$(1-b^2/a^2)^{1/2}$	Eccentricity $(1-c^2/b^2)^{1/2}$	θ_4	θ_5 or θ_{10}	θ_6	θ_7	θ_8	θ_{11}
0.2	0.2	2.999	−0.749	−3.000	6	−2.998	0.250
	0.4	2.994	−0.746	−3.002	6	−2.984	0.248
	0.6	2.969	−0.729	−3.005	6	−2.908	0.237
	0.8	2.891	−0.658	−2.958	6	−2.567	0.193
0.4	0.2	2.993	−0.746	−3.004	6	−2.984	0.248
	0.4	2.985	−0.741	−3.008	6	−2.962	0.244
	0.6	2.956	−0.720	−3.011	6	−2.873	0.232
	0.8	2.877	−0.646	−2.955	6	−2.511	0.186
0.6	0.2	2.960	−0.727	−3.031	6	−2.915	0.237
	0.4	2.946	−0.718	−3.040	6	−2.881	0.232
	0.6	2.908	−0.692	−3.048	6	−2.768	0.216
	0.8	2.828	−0.612	−2.985	6	−2.378	0.168
0.8	0.2	2.796	−0.642	−3.215	6	−2.634	0.193
	0.4	2.770	−0.628	−3.243	6	−2.584	0.186
	0.6	2.717	−0.594	−3.277	6	−2.449	0.168
	0.8	2.637	−0.513	−3.221	6	−2.053	0.124

Table 3. Sign of the coefficients in Hand's equations for a dilute suspension of rigid ellipsoids.

Positive	σ_4	σ_6	σ_7	σ_{11}	θ_4	θ_7	θ_{11}
Negative	σ_5	σ_8	σ_{10}	θ_5	θ_6	θ_8	θ_{10}

(i) *Almost spherical ellipsoids.* Let $a^2 = 1 + \varepsilon$, $b^2 = 1$, $c^2 = (1 + \varepsilon)^{-1}$, with $\varepsilon \ll 1$. Then, after some lengthy calculations, the asymptotic forms for the σ's are found to be:

$$\sigma_4 = 8.143[1 + O(\varepsilon^2)]\mu_0\varphi, \qquad \sigma_5 = \sigma_{10} = -2.543[1 + O(\varepsilon^2)]\mu_0\varphi,$$
$$\sigma_6 = 2\mu_0 + 5.255[1 + O(\varepsilon^2)]\mu_0\varphi, \qquad \sigma_7 = 1.633[1 + O(\varepsilon^2)]\mu_0\varphi,$$
$$\sigma_8 = -1.888[1 + O(\varepsilon^2)]\mu_0\varphi, \qquad \sigma_{11} = 0.843[1 + O(\varepsilon^2)]\mu_0\varphi.$$

Interestingly enough, the above result shows that several terms in Hand's equation will contribute to Einstein's (1906) formula, namely $P_{ij} = -p\delta_{ij} + 2\mu_0(1 + \frac{5}{2}\varphi)e_{ij}$, when ε goes to zero. As for the coefficients appearing in the differential equation for A_{ij}, these can be easily evaluated to yield:

$$\theta_{11} = 3 + O(\varepsilon^2), \quad \theta_5 = \theta_{10} = -\tfrac{3}{4} + O(\varepsilon^2), \quad \theta_6 = 3 + O(\varepsilon^2), \quad \theta_8 = 3 + O(\varepsilon^2), \quad \theta_{11} = \tfrac{1}{4} + O(\varepsilon^2).$$

Again, the solid body rotation equation of a sphere will be recovered through the contributions of several terms in (9).

(ii) *Slender ellipsoids.* One semi-diameter, a, is assumed to be much larger than the other two, and thus $b/a \ll 1$ and $c/a \ll 1$, corresponding to $a \gg 1$. The limiting form of the coefficients of the stress equation are then

$$\sigma_4 = \frac{a^3(b^2+c^2)^2}{\log\dfrac{4a}{b+c} - \dfrac{3}{2}}\mu_0\varphi(1+\chi), \qquad \sigma_5 = \sigma_{10} = -\frac{a(b^2+c^2)}{\log\dfrac{4a}{b+c} - \dfrac{3}{2}}\mu_0\varphi(1+\chi),$$

$$\sigma_6 = 2\mu_0\left\{1 + \varphi\frac{(b+c)(b^3+c^3)}{bc(b^2+c^2)}[1+O(\varepsilon)]\right\}, \qquad \sigma_7 = \frac{4}{a}\mu_0\varphi[1+O(\varepsilon)],$$

$$\sigma_8 = -\frac{8\mu_0\varphi}{a^3(b^2+c^2)}[1+O(\varepsilon)], \qquad \sigma_{11} = \frac{\mu_0\varphi}{a\left(\log\dfrac{4a}{b+c} - \dfrac{3}{2}\right)}(1+\chi),$$

where χ is $O(\varepsilon)$, and

$$\varepsilon = \left|\frac{b^2-c^2}{a^2}\right| \ln\left|\frac{b^2-c^2}{a^2}\right|.$$

The coefficients for the second of Hand's equations can be similarly evaluated:

$$\theta_4 = 2\left[\frac{(b^2+c^2)^2 + (bc)^2}{(b^2+c^2)}\right][1+O(\varepsilon')], \qquad \theta_5 = \theta_{10} = -\frac{2}{a^2}[1+O(\varepsilon')],$$

$$\theta_6 = \frac{-4}{b^2+c^2}[1+O(\varepsilon')], \qquad \theta_8 = \frac{-8}{a^2(b^2+c^2)}[1+O(\varepsilon')],$$

$$\theta_{11} = \frac{2}{a^4(b^2+c^2)}[1+O(\varepsilon')],$$

where

$$\varepsilon' = \max\left(\frac{1}{a^3}, \frac{b^2+c^2}{a^2}\right).$$

The stress relation for this suspension of slender particles follows readily:

$$P_{ij} = -p\delta_{ij} + 2\mu_0 e_{ij} + \frac{\mu_0\varphi a^3}{\ln 2a^{3/2} - \frac{3}{2}}\left[e_{lm}\overset{(a)(a)}{n_l n_m} + O(G\varepsilon)\right]\left[\tfrac{2}{3}\overset{(a)(a)}{n_i n_j} - \tfrac{1}{3}\overset{(b)(b)}{n_i n_j} - \tfrac{1}{3}\overset{(c)(c)(c)}{n_i n_j}\right]$$

and is identical to that derived by Batchelor (1970b) for the particular case of an extensional flow.

A couple of observations are in order regarding this result. First, the contribution to the bulk stress due to the inclusions is essentially felt through the terms $Sd(A_{ij})$ and $Sd(A_{il}A_{lj})$. Furthermore, the particle stress is multiplied by $a^3/(2\ln 2a^{3/2} - 3)$, a factor which can become quite large, since a was assumed much greater than unity. For the analysis to be valid, the volume concentration of particles must therefore be very small: $\varphi \ll (2/a^3)\ln 2a^{3/2}$. As was pointed out by Batchelor (1970b), this implies that the theory of dilute suspensions does not really apply to a suspension of slender particles, or, conversely, that the addition of even a small number of slender particles can exert a profound effect on the bulk stress of a fluid.

(iii) *Disk-like ellipsoids.* Let $a^2 = 1 + e/\varepsilon$, $b^2 = 1/\varepsilon(1+e)$, $c = \varepsilon$, with $\varepsilon \ll 1$ and $e \ll 1$. In this case, a and b are both large and almost equal. The asymptotic forms of the

σ's thus become:

$$\sigma_4 = \mu_0 \varphi \frac{11}{6\pi} \varepsilon^{-7/2} \{1 + \chi + O(\varepsilon^6)\},$$

$$\sigma_5 = \sigma_{10} = -\mu_0 \varphi \frac{11}{6\pi} \varepsilon^{-3/2} \left\{1 + \chi - \varepsilon^3 \left[1 + \chi + \frac{12}{11}(1 + \chi_1)\right] + O(\varepsilon^6)\right\},$$

$$\sigma_6 = 2\mu_0 \left\{1 + \varphi \frac{8}{3\pi} \varepsilon^{-3/2} \left[1 - \frac{8}{\pi} \varepsilon^{3/2} + O(e^2) + O\left(\frac{\varepsilon^3}{e}\right)\right]\right\},$$

$$\sigma_7 = \mu_0 \varphi \frac{3\pi}{4} \varepsilon^{1/2} \left[1 + O(e) + O\left(\frac{\varepsilon^{3/2}}{e}\right)\right],$$

$$\sigma_8 = -\mu_0 \varphi \frac{3\pi}{4} \varepsilon^{5/2} \left[1 + O(e) + O\left(\frac{\varepsilon^{3/2}}{e}\right)\right],$$

$$\sigma_{11} = \mu_0 \varphi \frac{11}{6\pi} \varepsilon^{1/2} \left\{1 + \chi - 2\varepsilon^3 \left[1 + \chi + \frac{12}{11}(1 + \chi_1)\right] + O(\varepsilon^6)\right\},$$

where
$$\chi = \frac{6\pi\varepsilon^{3/2}}{11e^2}(A'' + B'' - 2C'')[1 + 4\varepsilon^3 + O(\varepsilon^6)],$$

and
$$\chi_1 = \frac{\pi\varepsilon^{3/2}}{2e}(A'' - B'') - 1.$$

Both χ and χ_1 are $O[\max(e, e^{3/2}/e)]$ at most.

The asymptotic form of the coefficients θ can be similarly obtained:

$$\theta_4 = \varepsilon^{-1}[1 + O(\varepsilon^3) + O(e^2)], \qquad \theta_5 = \theta_{10} = -\varepsilon[1 + O(\varepsilon^3) + O(e^2)],$$

$$\theta_6 = -6\varepsilon[1 + O(\varepsilon^3) + O(e^2)], \qquad \theta_8 = -8\varepsilon^2[1 + O(\varepsilon^2)], \qquad \theta_{11} = \varepsilon^3[1 + O(\varepsilon^3) + O(e^2)].$$

The constitutive equation of such a suspension is then:

$$P_{ij} = -p\delta_{ij} + 2\mu_0 e_{ij} + \frac{2}{3\pi}\mu_0 \varphi \varepsilon^{-3/2} \left\{8e_{ij} + \left[\overset{(a)(a)}{n_i n_j} + \overset{(b)(b)}{n_i n_j} - 2\overset{(c)(c)}{n_i n_j}\right]\right.$$
$$\left.\left[e_{lm}\overset{(c)(c)}{n_l n_m} + O(Ge)\right]\right\}\left[1 + O(e) + O\left(\frac{\varepsilon^{3/2}}{e}\right)\right].$$

From this result, it is seen that the contribution to the stress due to the presence of the particles, is again of order a^3. Furthermore, since the denominator no longer contains a logarithmic term, as was the case with the corresponding expression for slender ellipsoids, it would appear that the effect of disk-like ellipsoids would be felt somewhat more strongly than that of slender ellipsoids of equal length. Again, the above constitutive equation is quite limited in scope, since it applies to extremely dilute suspensions only. However, it suggests that the presence of a few disk-like particles can have truly dramatic effects on the stress of a fluid, in fact, even more so than in the previous case.

(b) *Solid spheres: the effect of inertia*

The rheology of a dilute suspension of rigid spheres was first studied by Einstein (1906) in the absence of inertial effects and hydrodynamic interactions between particles. Later analyses showed that the suspension remained Newtonian and that the presence of the spheres was felt only through an increase in the shear viscosity. Recently, the influence of inertial forces on the steady-state rheology of such a suspension was examined by Lin, Peery & Schowalter (1970), who developed a regular perturbation solution to the Navier–Stokes equations for flow past a freely suspended sphere, when the Reynolds number R of the motion is small compared with unity, and showed that, as expected, the symmetry of the flow in the vicinity of a particle is destroyed by the presence of small inertia forces. Since this last result suggests the formation of a local anistropy, it is conceivable that a theory of anisotropic fluids, for example, a generalization of Hand's equations to include higher-order tensors, could describe the rheology of a dilute suspension of solid spheres, when R is not zero. Unfortunately, owing to the complexity of the problem, a general study of the rheology of such a suspension which includes time-dependent effects is presently not available, and hence the validity of the above hypothesis cannot be directly established. We shall proceed with this point of view, however, and explore to what extent the results obtained previously for dilute suspensions of deformable particles, can be used to generalize the rheological equation developed by Lin *et al.* Thus, we shall assume that for a general time-dependent linear shear field, the rheological equation for the suspension of solid spheres conforms to [8] and [9] with the addition, perhaps, of some higher-order tensors. Here the non-Newtonian behavior is obviously caused by the inertia forces, so that the time derivatives appearing in the constitutive equation should be multiplied by some measure of the inertia effect which, from fluid mechanical considerations, should in all likelihood be proportional to an algebraic power of the Reynolds number. Also, since R is assumed to be small compared with unity, for weakly time-dependent flows the same treatment used in conjunction with suspensions of elastic spheres and liquid droplets should be applicable here. The differential equations describing the anisotropy should then be amenable to solution by successive approximations, thereby reducing the constitutive relation of the suspension of solid spheres to the Rivlin–Ericksen equation. Evidently, even if this proves to be the case, it will not mean necessarily that a generalized theory of anisotropic fluids does describe a suspension of solid spheres in the presence of inertia effects, but rather that there is no contradiction between our hypothesis and the analytical results presently available.

We recall now that Lin, Peery & Schowalter (1970) derived their constitutive equation for the special case of a steady simple shear flow in the x_1 direction: $u_1 = Gx_2$. They obtained for the bulk stress to $O(R^{3/2})$:

$$P_{11} = -p + \mu_0\varphi GR[-\tfrac{2}{3} + R^{1/2}(30B_{-2} - 5B_0)],$$

$$P_{22} = -p + \mu_0\varphi GR[\tfrac{2}{3} + R^{1/2}(-30B_{-2} - 5B_0)], \quad P_{33} = -p + 10\mu_0\varphi GR^{3/2}B_0,$$

$$P_{12} = P_{21} = \mu_0(1 + \tfrac{5}{2}\varphi)G + 30\mu_0\varphi GR^{3/2}B_2, \qquad P_{23} = P_{32} = P_{31} = P_{13} = 0, \quad [27]$$

where B_{-2}, B_0, B_2 are numerical constants. The Reynolds number of the motion is based upon the particle radius, a, the magnitude of the shear rate G, the viscosity μ_0, and the density, ρ, of the suspending fluid, i.e. $R = a^2G\rho/\mu_0$. We shall show now how the above constitutive relation can be considered as a particular case of the Rivlin–Ericksen equation, evaluated for a steady simple shear flow.

Unfortunately, even under these special flow conditions, [2] contains several unknown coefficients, α_k, which cannot be determined uniquely from the limited information provided by [27]. However, since the latter was obtained from a regular perturbation solution, developed for small values of R, or equivalently of G, some conclusions can be drawn regarding the order of magnitude of the various α's. Thus, dividing through [2] by $\mu_0 G$ yields

$$\frac{P_{ij} - p\delta_{ij}}{\mu_0 G} = \frac{\alpha_1}{\mu_0}\frac{e_{ij}}{G} + \frac{\alpha_2}{\mu_0}\frac{1}{G}\frac{\mathscr{D}e_{ij}}{\mathscr{D}t} + \frac{\alpha_3}{\mu_0}\frac{1}{G}Sd(e_{il}e_{lj}) + \frac{\alpha_4}{\mu_0}\frac{1}{G}Sd\left(e_{il}\frac{\mathscr{D}e_{lj}}{\mathscr{D}t}\right) + \frac{\alpha_5}{\mu_0}\frac{1}{G}\frac{\mathscr{D}^2 e_{ij}}{\mathscr{D}t^2} + \frac{\alpha_6}{\mu_0}\frac{1}{G}Sd\left(\frac{\mathscr{D}e_{il}}{\mathscr{D}t}\frac{\mathscr{D}e_{lj}}{\mathscr{D}t}\right) + \cdots, \qquad [28]$$

in which every term is dimensionless. Consequently, since $\mathscr{D}e_{ij}/\mathscr{D}t$, $Sd(e_{il}e_{lj})$, etc. are proportional to G^2 at least, the α's must take the following forms:

$$\alpha_1 = \mu_0 f_1(R),\ \alpha_2 = (\mu_0/G)f_2(R),\ \alpha_3 = (\mu_0/G)f_3(R),\ \alpha_4 = (\mu_0/G^2)f_4(R),\ \alpha_5 = (\mu_0/G^2)f_5(R),$$
$$\alpha_6 = (\mu_0/G^3)f_6(R), \text{ etc.,}$$

where the $f_k(R)$'s represent unknown functions of the Reynolds number. Recalling that the expression given by Schowalter *et al.* can be considered as an expansion for small values of G, we require that, as $R \to 0$, the coefficients of the higher-order terms in G be smaller than those of lower-order, i.e.

$$\lim_{R\to 0}\left(\frac{f_4 \text{ or } f_5}{f_2 \text{ or } f_3}\right) = 0, \qquad \lim_{R\to 0}\left(\frac{f_6 \text{ or } f_7}{f_4 \text{ or } f_5}\right) = 0, \text{ etc.} \qquad [29]$$

By comparing [27] and [28] evaluated for a simple shear flow, it becomes apparent that the term $-5B_0\mu_0\varphi GR^{3/2}$ appearing in both P_{11} and P_{22}, as given by [27] can arise either from $\alpha_3 Sd(e_{il}e_{lj})$ or from a higher-order term, (e.g. $\alpha_6 Sd(\mathscr{D}e_{il}/\mathscr{D}t)(\mathscr{D}e_{lj}/\mathscr{D}t)$). Consequently, there are three possible cases:

$$f_3(R) = O(R^{3/2}) \quad \text{and} \quad f_6(R) = O(R^{3/2}),$$

or $$f_3(R) = O(R^n) \quad \text{and} \quad f_6(R) = O(R^{3/2}), n > 3/2,$$

or $$f_3(R) = O(R^{3/2}) \quad \text{and} \quad f_6(R) = O(R^n), n > 3/2,$$

of which only the last combination is compatible with condition [29]. It follows, therefore, that, for $k \geq 4$, $f_k(R) = O(R^n)$, $n > 3/2$. Thus, since [27] contains terms which are at most $O(R^{3/2})$, to this order of approximation, [2] reduces to:

$$P_{ij} = -p\delta_{ij} + \alpha_1 e_{ij} + \alpha_2(\mathscr{D}e_{ij}/\mathscr{D}t) + \alpha_3 Sd(e_{il}e_{lj}) + O(GR^{3/2}). \qquad [30]$$

Of course, there are infinitely many ways of choosing the functional dependence of the

α's on the invariants of e_{ij} and its time derivatives. By analogy with the results obtained for suspensions of deformable particles, however, it seems reasonable to assume that, again to this order of approximation, the remaining coefficients depend on the invariants of e_{ij} only, the invariants of $\mathscr{D}e_{ij}/\mathscr{D}t$ being at least of order R^4. Thus, since e_{ij} is proportional to R, a power of a given invariant of e_{ij} will give rise to a corresponding power of R. For example, $(e_{lm}e_{lm})^{1/2}$ will be proportional to R, whereas $(e_{lm}e_{mp}e_{pl})^{1/6}$ will be proportional to $R^{1/2}$, etc. Thus, by taking into account the fact that the expression derived by Lin *et al.* is valid to order $(R^{3/2})$, we let:

$$\alpha_1 = \mu_0[\delta_0 + \delta_1(2e_{lm}e_{lm})^{1/4} + \delta_2(2e_{lm}e_{lm})^{3/4} + \delta_3(e_{lm}e_{mp}e_{pl})^{1/6} + \delta_4(e_{lm}e_{mp}e_{pl})^{1/2}],$$

$$\alpha_2 = \mu_0/G[\delta_5 + \delta_6(2e_{lm}e_{lm})^{1/4} + \delta_7(e_{lm}e_{mp}e_{pl})^{1/6}],$$

$$\alpha_3 = \mu_0/G[\delta_8 + \delta_9(2e_{lm}e_{lm})^{1/4} + \delta_{10}(2e_{lm}e_{mp}e_{pl})^{1/6}],$$

where the δ's are at most $O(1)$. This particular choice for $\alpha_1, \alpha_2, \alpha_3$ was guided by the form of [27] and by the results of the previous section and, although quite arbitrary, appears reasonable.

Equation [30] is now evaluated for the shear flow considered by Lin *et al.* After noting that R is proportional to G, it is then possible to compare the stress components as predicted by [27] and [30]. It follows readily that

$$\delta_0 = 2(1 + \tfrac{5}{2}\varphi), \qquad \delta_1 = 0, \qquad \delta_2 = 60\varphi B_2(R/G)^{3/2}, \qquad \delta_5 = \tfrac{8}{3}\varphi R,$$

$$\delta_6 = 120\varphi B_{-2}(R^{3/2}/G^{1/2}), \qquad \delta_8 = 0, \qquad \delta_9 = -60\varphi B_0(R^{3/2}/G^{1/2}).$$

Evidently, the coefficients δ_4, δ_7, and δ_{10} cannot be determined, because, for a simple shear, the product $e_{lm}e_{mp}e_{pl}$ is identically zero. However, since the coefficient, δ_1, of $e_{ij}(2e_{lm}e_{lm})^{1/4}$ was found to be zero, it is reasonable to assume that δ_3, which multiplies $e_{ij}(e_{lm}e_{mp}e_{pl})^{1/6}$, will also vanish.

Therefore, after replacing B_0, B_2, and B_{-2} by their values, we finally obtain as a possible constitutive equation for a suspension of solid spheres, subjected to small inertial forces,

$$P_{ij} = -p\delta_{ij} + 2\mu_0 e_{ij} + \varphi\mu_0\left\{2e_{ij}\left[\frac{5}{2} + 1{\cdot}344R^{3/2}\left(\frac{2e_{lm}e_{lm}}{G^2}\right)^{3/4} + \delta_4 R^{3/2}\left(\frac{e_{lm}e_{mp}e_{pl}}{G^3}\right)^{1/2}\right]\right.$$
$$+ \frac{1}{G}\frac{\mathscr{D}e_{ij}}{\mathscr{D}t}\left[\frac{8}{G}R - 0{\cdot}575R^{3/2}\left(\frac{2e_{lm}e_{lm}}{G^2}\right)^{1/4} + \delta_7 R^{3/2}\left(\frac{e_{lm}e_{mp}e_{pl}}{G^3}\right)^{1/6}\right]$$
$$\left.+ \frac{1}{G}Sd(e_{il}e_{lj})\left[-0{\cdot}434R^{3/2}\left(\frac{2e_{lm}e_{lm}}{G^2}\right)^{1/4} + \delta_{10}R^{3/2}\left(\frac{e_{lm}e_{mp}e_{pl}}{G^3}\right)^{1/6}\right] + O(GR^{3/2})\right\}.$$

This above development is of interest because it demonstrates how a plausible general form for the constitutive equation of a given fluid can be inferred from a solution that was developed only for very special flow conditions. Of course, as was remarked earlier, the fact that the above is a particular case of the Rivlin–Ericksen equation does not imply that this suspension of solid spheres is an anisotropic fluid. Rather, the important point is that, by taking this somewhat unorthodox approach to the problem, we have been able to generalize the results given by Lin *et al.*, in a fashion consistent with earlier results of suspension theory.

CONCLUSION

From this comparison between the continuum and the phenomenological approaches to non-Newtonian fluid mechanics, it is now possible to draw several conclusions. The most obvious one perhaps, is that the theory of suspensions allows us to form some judgment regarding the physical significance and potential usefulness of the various phenomenological constitutive relations. In particular, it has been shown here that Oldroyd's equations do not really represent the type of non-Newtonian behavior exhibited by a dilute suspension of elastic particles. As for the Rivlin–Ericksen equation, it was found to model a dilute suspension, but only under the quite restrictive condition of weakly time-dependent or steady flows. This theory thus appears to have some physical basis, but in a rather limited sense. By contrast, it is very clear that Hand's theory of anisotropic fluids, or its generalization to more complex anisotropics, depicts realistically the properties of those non-Newtonian fluids which contain some anisotropy on the microscopic scale, and it is worthwhile to note that the exact analytical treatment of suspensions does yield equations which are of the very form predicted by Hand. Furthermore, Hand's relations are not restricted to truly anisotropic fluids, but are also adequate for substances which become anisotropic only under motion.

The experience gained through the study of suspensions of deformable particles has thus provided new information regarding the physical significance of the various phenomenological equations and of the unknown coefficients they contain. In turn, this new knowledge has made possible a plausible extension to a general linear shear flow of the stress expression obtained by Lin *et al.* for a dilute suspension of solid spheres, subjected to a simple shearing motion, in the presence of small inertia forces.

Some further remarks should also be made regarding the choice of the anisotropy tensor, A_{ij}, in Hand's theory, since the value of the various coefficients, σ, in [8] depends on the definition of A_{ij}. Thus, if the anisotropy, instead of being described by A_{ij}, is represented by a tensor A'_{ij}, such that $A_{ij} = yA'_{ij} + x\delta_{ij}$, where x and y are arbitrary scalar quantities, then the coefficients σ'_k corresponding to [8] with A_{ij} replaced by A'_{ij}, are linked to the σ_k by means of:

$$\sigma_3 = \sigma'_3/y, \qquad \sigma_4 = [\sigma'_4 - (2x/y)(\sigma'_5 + \sigma'_{10}) + (4x^2/y^2)\sigma'_{11}]y^{-2},$$
$$\sigma_5 = [\sigma'_5 - (2x/y)\sigma_{11}]/y^3, \qquad \sigma_6 = \sigma'_6 - (x/y)\sigma'_7 + (x^2/y^2)\sigma'_8,$$
$$\sigma_7 = [\sigma'_7 - (2x/y)\sigma'_8]/y, \qquad \sigma_8 = \sigma'_8/y,$$
$$\sigma_9 = \sigma'_9/y^2, \qquad \sigma_{10} = (\sigma'_{10} - (2x/y)\sigma'_{11})/y^3,$$
$$\sigma_{11} = \sigma'_{11}/y^4. \qquad [31]$$

Table 4. Sign of the coefficients in [8] for suspensions of droplets and deformable particles.

≥ 0	σ_3	σ_4	σ_6	σ_7
≤ 0	σ_8			

We recall, now, that in studying the suspensions of deformable particles, the tensor A_{ij} was chosen to represent the non-sphericity, i.e. the deviation from isotropy of the particle, whereas, in the case of solid ellipsoids, A_{ij} stood for the full tensor defining the ellipsoidal shape. Therefore, before drawing some global conclusions regarding the phenomenological coefficients, σ_k, and in particular their sign, it is necessary to choose a consistent representation for A_{ij}. Thus, for the suspensions of droplets and of elastic spheres, the anisotropy will be represented, respectively, by $A_{ij} = \delta_{ij} + F_{ij}$, and by $A_{ij} = \delta_{ij} + C_{ij}$ corresponding to $x = 1$ and $y = 1$ in [31]. Also the σ_k in [17] and [24] should now be read as σ'_k. The new coefficients σ can then easily be computed from [8], [17] and [24], but their expression is not given here since it is of rather limited interest. However, it is important to note that their sign, given in Table 4, will be the same as the sign of the corresponding σ'_k's.

No conclusion can be drawn for σ_9, as given by [17], since its sign depends on the value of the viscosity ratio λ. In contrast, from a comparison of tables 3 and 4, it can easily be established that σ_3, σ_4, σ_6, σ_7 are all positive or zero, whereas σ_8 is negative or zero for suspensions of rigid or deformable particles alike. This result might be more general and has some interesting implications.

Unfortunately, it is not possible to draw such a conclusion regarding the coefficients appearing in Hand's second equation, since, for an emulsion, their sign depends on the value of the viscosity ratio. This might be explained by the fact that this equation describes essentially the motion of an individual particle, which is known to differ in a fundamental way in the three cases considered. Similarly, although this would be of importance to the subject of rheology, one cannot draw any conclusions regarding the sign of the ratio $(P_{22} - P_{33})/(P_{11} - P_{22})$ when the bulk velocity is of the form $u_i = \delta_{i1}x_2$.

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Sommaire—Les résultats théoriques du domaine de la rhéologie de suspensions sont étudiés dans le contexte général de mécanique de continuité non-linéaire, afin d'extraire des renseignements concernant la formulation d'une relation entre le modèles de contrainte phénoménologique et des fluides non-Newtoniens. Les équations constitutives obtenues pour des suspensions diluées d'ellipsoïdes solides, de sphères élastiques et de gouttelettes liquides sont montrées être conformes à un modèle phénomènologique simple déjà proposé par Hand, et qui apparaît être une équation de fondement physique considérable et d'une utilité indispensable. De plus dans les trois cas ci-dessus, certains des coefficients de l'équation de Hand conservent le même signe. Ce résultat est d'application plus générale et peut avoir des implications intéressantes.

Auszug—Es werden theoretische Ergebnisse auf dem Gebiet der Suspensionsrheologie im allgemeinen Zusammenhang nichtlinearer Kontinuummechanik untersucht, um Information in Bezug auf Bildung eines phänomenologischen Spannungsverhältnisses zu gewinnen, um nicht-Newton'sche Flüssigkeiten zu bilden. Es wird gezeigt, daß die grundlegenden Gleichungen, die für verdünnte Suspensionen fester Ellipsoide, elastischer Kugeln und flüssiger Tröpfchen erhalten wurden, mit einem einzigen, von Hand zuerst vorgeschlagenen, phänomenologischen Modell übereinstimmen, was sich als eine Gleichung mit bedeutender physikalischer Grundlage und möglicher Nützlichkeit entwickelt. Außerdem wird gefunden, daß in allen drei obigen Fällen bestimmte Koeffizienten in Hand's Gleichung das gleiche Zeichen beibehalten. Dieses Ergebnis könnte allgemeiner zutreffen und möglicherweise interessante Implikationen haben.

Резюме—Рассматриваются теоретические результаты в области реологии суспензии в смысле нелинейной механики сплошной среды, чтобы получить информацию относительно формулировки феноменологического напряжения по отношению к моделям неньютоновских текучих сред. Уравнения полученные для разбавленных суспензий сплошных эллипсоидов, для эластических сфер и для капалек жидкости оказалось соответствуют одой феноменологической модели, первоначально предложенной Хандом, которая является уравнением с существенным физическим основанием и потенциальной приемлемостью. Кроме того, во всех трех случаях приведенных выше определенные коэффициенты в уравнениях Ханда оказвывается сохраняют один и тот же символ. Результаты можно применять более общирно и они имеют интересное значение.